JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 51, 493–515 (1975)

Asymptotic Simplification and Factorization of Linear Partial Differential Equations

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1. INTRODUCTION

Asymptotic methods for the solution of initial and boundary value problems for linear partial differential equations have undergone much development in recent years. They have been used for problems which involve a large parameter or where the solutions of the equations are discontinuous or singular because the data for the problem are of that form. In the latter case, when the equations are of hyperbolic type, the asymptotic solutions are generally referred to as progressing wave expansions.

In previous work [15] we have considered the relationship between the aforementioned asymptotic methods and the asymptotic theories developed for the solution of ordinary differential equations with a large parameter and turning points. On that basis, formal uniform progressing wave expansions were constructed for certain second-order partial differential equations of mixed type in two independent variables. Such uniform expansions also occur in the asymptotic theory of partial differential equations in more than two variables and had been considered earlier, for various problems, by Ludwig [9] and Kravtsov [7], among others. (General references for asymptotic methods of the type considered here are the texts by Courant [1], Wasow [13], and Feshchenko *et al.* [3], and the forthcoming book by Keller and Lewis [17].)

In this paper we consider the formal simplification and solution of linear first-order systems and single higher-order equations in two independent variables. We show that a factorization of the characteristic form of the given partial differential equation or system into two mutually prime factors induces a formal block diagonalization of the system of equations and a formal factorization of the single equation into a product of two lower-order equations. Depending on the number of mutually prime factors occurring in the characteristic form of the given system or higher-order equation, the simplification or factorization process may be reapplied to the reduced system or factored equation. The techniques used are patterned after those developed by Sibuya and Wasow [13] and by Langer [8] and Erdelyi [2] for the asymptotic simplification and solution of first-order systems and single higher-order ordinary differential equations with a large parameter, respectively. These results are obtained in Sections 2 and 3.

The effect of our formal simplification or factorization method is to reduce the asymptotic theory for the given problem to that for two or more simpler problems each involving systems with fewer dependent variables or single equations of lower order. If asymptotic theories for the reduced systems or equations are known, further results can be obtained. For example, if a reduced system contains only two dependent variables, our previously obtained uniform progressing wave expansions for such problems can be used if certain conditions are met [15]. In general, as is the case for similar asymptotic theories for ordinary differential equations, separate investigations are necessary for the variety of reduced problems that result from our method in order to achieve further asymptotic simplification.

Additionally, in Section 4, we consider reduced equations which correspond to characteristics that are simple or have constant multiplicity. While in such cases conventional progressing wave expansions can be constructed directly for the given equations, we consider instead special solution forms for the reduced equations. They result in a new expansion form for the given equation, which has the property that the usual progressing wave expansion can be retrieved from it. In particular, on considering highly oscillatory solutions of the given equations, inadequacies in the conventional expansion may arise in certain cases because of the presence of "secular terms." These difficulties may be circumvented by an appropriate use of the new expansion form referred to above.

For hyperbolic partial differential equations and systems in any number of independent variables, progressing wave expansions can be constructed corresponding to simple or uniformly multiple characteristics. Considering first-order hyperbolic systems with more than two independent variables, we present a direct method for the construction of the new expansion form considered in Section 4. This result may be thought of as a partial extension of the simplification theory as developed for equations with two independent variables to more general problems but only for the case of simple or uniformly multiple characteristics. Also, for certain problems, as indicated above, the new expansion has greater validity. These topics are discussed in Section 5.

In Section 6, a special example, the one-dimensional Klein–Gordon equation, is examined. It is expressed both as a single second-order equation as well as a first-order system. We show how secular terms arise in the conventional progressing wave solutions and how, by using the methods of

Sections 2–5, this inadequacy in the standard expansion form can be removed.

In the concluding section, the relationship of the methods considered here to those used in other work, as well as their possible further application to nonlinear problems, will be touched upon.

2. Formal Simplification of First-Order Systems

We consider the first-order linear system of partial differential equations in two independent variables.

$$u_t + A(x, t)\, u_x + B(x, t)\, u = 0, \tag{1}$$

where $u(x, t)$ is an n-vector with components $u^i(x, t)$ $[i = 1,..., n]$ and $A(x, t)$ and $B(x, t)$ are real-valued n by n matrices.

The characteristic curves $\varphi(x, t) =$ constant of (1), as well as their multiplicities, are determined from the eigenvalues $\lambda(x, t)$ of the matrix $A(x, t)$, and their multiplicities, via the equation

$$\lambda(x, t)\, \varphi_x + \varphi_t = 0. \tag{2}$$

Given a domain D in the (x, t)-plane, we assume the characteristic polynomial of the matrix $A(x, t)$,

$$p(x, t, \lambda) = \det[A(x; t) - \lambda(x, t)\, I], \tag{3}$$

where I is the n by n identity matrix, admits the factorization

$$p(x, t, \lambda) = p_1(x, t, \lambda)\, p_2(x, t, \lambda), \tag{4}$$

where p_1 and p_2 are polynomials in λ of degree m and k, respectively, with $m + k = n$. The polynomials $p_1(x, t, \lambda)$ and $p_2(x, t, \lambda)$ are assumed to have no common roots in D, i.e., they are mutually prime. The eigenvalues $\lambda_r(x, t)$ of $A(x, t)$ can therefore be arranged such that $\lambda_1 ,..., \lambda_m$ and $\lambda_{m+1} ,..., \lambda_n$ are the zeros of $p_1(x, t, \lambda)$ and $p_2(x, t, \lambda)$, respectively and $\lambda_i(x, t) \neq \lambda_j(x, t)$ for $i \leqslant m$ and $j > m$, for all points (x, t) in D. Within the set of zeros of p_1 and p_2, the λ_r may be simple roots or have constant or variable multiplicities involving two or more of the roots.

These properties of the eigenvalues $\lambda(x, t)$ carry over to their corresponding characteristics, resulting in classes of coincidence patterns for the characteristics in D. The formal simplification method, which we now apply to (1), reduces the asymptotic theory for the given system, to a study of two systems each having fewer dependent variables whose characteristics have coincidence patterns which are determined from the zeros of $p_1(x, t, \lambda)$ and $p_2(x, t, \lambda)$.

Following the methods used by Sibuya and Wasow [13] for ordinary differential equations, we may assume that the matrix $A(x, t)$ in (1) has the block-diagonal form

$$A(x, t) = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}, \tag{5}$$

where A_{11} and A_{22} are m by m and k by k matrices, respectively, and the eigenvalues of $A_{11}(x, t)$ and $A_{22}(x, t)$ are $\lambda_1, \ldots, \lambda_m$ and $\lambda_{m+1}, \ldots, \lambda_n$, respectively. This follows from a result of Sibuya [11], who proved that when $A(x, t)$ is a C^∞ (analytic) function of the variables x and t in a domain D, and the eigenvalues of $A(x, t)$ have the aforementioned properties in D, a similarity transformation, involving a C^∞ (analytic) matrix $T(x, t)$, exists which transforms $A(x, t)$ into block-diagonal form (as given in (5)) in the neighborhood of any point (x_0, t_0) in D. (See also Wasow [13, Chap. 7].) Under certain additional conditions, Sibuya [12] has extended this locally valid result into a global result. Given the matrix $T(x, t)$, the transformation

$$u(x, t) = T(x, t)\, w(x, t) \tag{6}$$

would take (1) into a system in $w(x, t)$, where the coefficient of w_x has the block-diagonal form (5), if $A(x, t)$ is not already assumed to be of that form. We also assume that $B(x, t)$ has the same differentiability properties as $A(x, t)$.

With $\partial_x \equiv \partial/\partial x$, let

$$u(x, t) = P(x, t, \partial_x)\, v(x, t), \tag{7}$$

where the n by n matrix $P(x, t, \partial_x)$ has the formal expansion

$$P(x, t, \partial_x) = \sum_{r=0}^{\infty} P_r(x, t)\, \partial_x^{-r}, \tag{8}$$

with the matrices $P_r(x, t)$ to be specified, so that the transformed system in $v(x, t)$ has a block-diagonal form similar to that of the principal part of (1). Setting

$$v_0 \equiv v; \qquad \partial_x[v_r] = v_{r-1}; \qquad r \geqslant 0 \tag{9}$$

we obtain

$$v_r(x, t) = \partial_x^{-1}[v_{r-1}] = \cdots = \partial_x^{-r}[v_0] \equiv \partial_x^{-r}[v] \tag{10}$$

on dropping all constants of integration. (This is valid if we assume $v(x, t)$ vanishes identically for all $x < x_0$, for some x_0, and take $-\infty$ as the lower limit in the integral operator ∂_x^{-1}.) We shall assume that the system (1) is of hyperbolic or of mixed type (having a domain of hyperbolicity, in the latter case), as the formal expansions we construct are meaningful for such equations. Using (10) the expansion (7) takes the form

$$u(x, t) = \sum_{r=0}^{\infty} P_r(x, t)\, v_r(x, t), \tag{11}$$

and it will be seen below for certain cases, with an appropriate choice of coordinates, that any discontinuities or singularities occurring in $v(x, t)$, resulting from data given at $t =$ constant, say, are smoothed out in the higher terms of the expansion.

Inserting (7) into (1) and multiplying across by P^{-1}, (P_0^{-1} will be shown to exist), we obtain

$$v_t + (P^{-1}AP)\, v_x + (P^{-1}P_t + P^{-1}AP_x + P^{-1}BP)\, v = 0. \tag{12}$$

The matrices P_t and P_x are defined as

$$\sum_{r=0}^{\infty} \frac{\partial P_r}{\partial t} \partial_x^{-r} \qquad \text{and} \qquad \sum_{r=0}^{\infty} \frac{\partial P_r}{\partial x} \partial_x{}^{r},$$

respectively. Let the matrix $C(x, t, \partial_x)$ be formally defined as

$$C(x, t, \partial_x) = \sum_{r=0}^{\infty} C_r(x, t)\, \partial_x^{-r}, \tag{13}$$

and set

$$C(x, t, \partial_x)\, \partial_x = (P^{-1}AP)\, \partial_x + P^{-1}P_t + P^{-1}AP_x + P^{-1}BP. \tag{14}$$

Multiplying across by $P(x, t, \partial_x)$ in (14) gives

$$(AP - PC)\, \partial_x = -P_t - AP_x - BP. \tag{15}$$

Using (8) and (14) we expand both sides of (15) in a series of the form $\sum_{r=0}^{\infty} H_r(x, t)\, \partial_x^{-r+1}$. In doing so, repeated integrations by part are necessary to bring the product $P(x, t, \partial_x)\, C(x, t, \partial_x)$ into the required form. We have

$$\begin{aligned} P(x, t, \partial_x)\, C(x, t, \partial_x)\, \partial_x &= \left(\sum_{r=0}^{\infty} P_r(x, t)\, \partial_x^{-r}\right) \left(\sum_{s=0}^{\infty} C_s(x, t)\, \partial_x^{-s+1}\right) \\ &= \sum_{r=0}^{\infty} \sum_{s=0}^{r} P_s \partial_x^{-s} C_{r-s} \partial_x^{s-r+1} \qquad (16) \\ &= P_0 C_0 \partial_x + \sum_{r=1}^{\infty} (P_0 C_r + P_r C_0)\, \partial_x^{-r+1} + \sum_{r=1}^{\infty} D_r \partial_x^{-r+1}, \end{aligned}$$

where D_r contains products of the matrices P_j and $\partial_x^k[C_i]$ ($k \geqslant 0$) with $i, j < r$. (As indicated above, all constants of integration are dropped in the integrations in (16).) Equating the coefficients of ∂_x^{-r+1} in (15) yields a recursive system of equations for the specification of the matrices P_r and C_r,

$$AP_0 - P_0C_0 = 0, \tag{17}$$

$$AP_r - P_rC_0 = P_0C_r + M_r; \qquad r \geqslant 1, \tag{18}$$

where the matrix $M_r(x, t)$ contains no terms P_j or $\partial_x^k[C_i]$ ($k \geqslant 0$) with $i, j \geqslant r$.

We set $C_0(x, t) = A(x, t)$ so that the equation satisfies by $v(x, t)$

$$v_t + C(x, t, \partial_x)\, v_x = v_t + Av_x + \sum_{r=0}^{\infty} C_{r+1}v_r = 0 \tag{19}$$

has the same principal part as (1). The remaining $C_r(x, t)$ ($r \geqslant 1$) are to be specified as to have a block-diagonal form similar to that of $A(x, t)$ (see (5)), thereby block-diagonalizing the entire system (19).

With $C_0 = A$, Eqs. (17) and (18) are identical in form to those which appear in the theory of asymptotic simplification of systems of ordinary linear differential equations with a large parameter (see Wasow [13, Chap. 7]). According to that theory, on setting $P_0 = I$, the $n \times n$ identity matrix, Eq. (18) can be solved with matrices $P_r(x, t)$ and $C_r(x, t)$ having the form

$$P_r = \begin{bmatrix} 0 & P_r^{12} \\ P_r^{21} & 0 \end{bmatrix}; \qquad C_r = \begin{bmatrix} C_r^{11} & 0 \\ 0 & C_r^{22} \end{bmatrix}, \tag{20}$$

where the diagonal blocks in P_r and C_r are m by m and k by k matrices, as in (5). Putting (20) into (18), we obtain four systems of equations

$$\begin{array}{ll} 0 = C_r^{11} + M_r^{11} & A_{11}P_r^{12} - P_r^{12}A_{22} = M_r^{12} \\ A_{22}P_r^{21} - P_r^{21}A_{11} = M_r^{21} & 0 = C_r^{22} + M_r^{22}, \end{array} \tag{21}$$

where M_r is split into four blocks corresponding to those in P_r and C_r. Following Wasow [13] we set $C_r^{11} = -M_r^{11}$ and $C_r^{22} = -M_r^{22}$. The matrices P_r^{12} and P_r^{21} are uniquely determined from the remaining equations in (21), since A_{11} and A_{22} have no common eigenvalues. Carried out recursively, this procedure completely specifies the matrices $P_r(x, t)$ and $C_r(x, t)$.

Each of the two diagonal blocks in the transformed system (19), yields a "reduced" system of equations (involving fewer dependent variables) with the principal part containing either the matrix $A_{11}(x, t)$ or $A_{22}(x, t)$. If the characteristics of either of the reduced systems can be divided into additional coincidence patterns, our method can again be applied to that system to obtain a further formal simplification. If a reduced system admits no further simplification because its characteristics all fall into a single coincidence pattern, i.e., the problem is "irreducible," a further simplification for the reduced problem can be obtained, if a related or comparison equation can be constructed for the problem. This will be considered in Section 4, where progressing wave expansions for certain coincidence patterns of the characteristics will be obtained.

3. Formal Factorization of Single Higher-Order Equations

We consider the partial differential operator of order n in two independent variables

$$L \equiv \sum_{s=0}^{n} p_s(x, t, \partial_x)\, \partial_x{}^s \partial_t^{n-s} \tag{22}$$

where $\partial_x \equiv \partial/\partial_x$, $\partial_t \equiv \partial/\partial_t$, $p_0(x, t, \partial_x) \equiv 1$, and

$$p_s(x, t, \partial_x) = \sum_{r=0}^{\infty} p_{sr}(x, t)\, \partial_x^{-r}; \qquad s = 0, 1, \ldots, n. \tag{23}$$

The principal part of L is defined as

$$L_p \equiv \sum_{s=0}^{n} p_{s0}(x, t)\, \partial_x{}^s \partial_t^{n-s}; \qquad p_{00}(x, t) \equiv 1, \tag{24}$$

and the equation satisfied by the characteristics $\varphi(x, t) = \text{constant}$ of $L[u(x, t)] = 0$ is

$$\sum_{s=0}^{n} p_{s0}(x, t)\, \varphi_x{}^s \varphi_t^{n-s} = 0. \tag{25}$$

Since $p_{00}(x, t) \equiv 1$ implies that $t = \text{constant}$ is noncharacteristic, we may define

$$\omega(x, t) = \varphi_t/\varphi_x \tag{26}$$

and bring the characteristic equation (25) into the form

$$\Omega(x, t, \omega) = \sum_{s=0}^{n} p_{s0}(x, t)\, \omega^{n-s} = 0. \tag{27}$$

It follows from (26) and (27) (compare with (2) and (3)), that the coincidence patterns of the characteristics of $L[u] = 0$, in a given domain D, are determined from the multiplicities of the zeros $\omega(x, t)$ of $\Omega(x, t, \omega)$.

Paralleling the approach in Section 2, we assume that the polynomial $\Omega(x, t, \omega)$ admits the factorization

$$\Omega(x, t, \omega) = \Omega_2(x, t, \omega)\, \Omega_1(x, t, \omega), \tag{28}$$

where Ω_1 and Ω_2 are mutually prime in D, and are given by

$$\Omega_1(x, t, \omega) = \sum_{i=0}^{m} q_{i0}(x, t)\, \omega^{m-i}; \qquad q_{00}(x, t) \equiv 1, \tag{29}$$

$$\Omega_2(x, t, \omega) = \sum_{j=0}^{k} \alpha_{j0}(x, t)\, \omega^{k-j}; \qquad \alpha_{00}(x, t) \equiv 1, \tag{30}$$

with $m + k = n$. The Eqs. (28)–(30) imply that

$$p_{s0}(x, t) = \sum_{i+j=s} q_{i0}(x, t)\, \alpha_{j0}(x, t). \tag{31}$$

Corresponding to the factorization of the characteristic equation of $L[u] = 0$, we now construct a factorization of the differential operator L in the form

$$L = L_2 L_1 \equiv \left[\sum_{j=0}^{k} \alpha_j(x, t, \partial_x)\, \partial_x{}^j \partial_t^{k-j}\right] \left[\sum_{i=0}^{m} q_i(x, t, \partial_x)\, \partial_x{}^i \partial_t^{m-i}\right] \tag{32}$$

where the α_j and q_i admit the formal expansions

$$\alpha_j(x, t, \partial_x) = \sum_{r=0}^{\infty} \alpha_{jr}(x, t)\, \partial_x^{-r} \tag{33}$$

$$q_i(x, t, \partial_x) = \sum_{r=0}^{\infty} q_{ir}(x, t)\, \partial_x^{-r}, \tag{34}$$

with the α_{jr} and q_{ir}, for $r > 0$, to be specified and the α_{j0} and q_{i0} given as in the above. Our problem is now identical in form to that considered by Erdelyi [2] in his asymptotic factorization theory for nth-order linear ordinary differential equations with a large parameter, apart from the presence of the differential operator ∂_x rather than a parameter k.

To specify the α_j and q_i in (32), we first equate the coefficients of the operators $\partial_t{}^r$ $(r = 0, 1,..., n)$ in L and in $L_2 L_1$. Since the operators ∂_x and ∂_t commute, we may use Erdelyi's [2] results for the ordinary differential equations problem and obtain

$$\sum_{i+j+\ell=s} \binom{k-j}{\ell} [\alpha_j \partial_x{}^j]\, [q_i^{(\ell)} \partial_x{}^i] = p_s \partial_x{}^s; \qquad s = 0, 1,..., n, \tag{35}$$

where $\binom{k-j}{\ell}$ is the binomial coefficient and

$$q_i^{(l)} = \sum_{r=0}^{\infty} \frac{\partial^\ell q_{ir}(x, t)}{\partial t^\ell} \partial_x^{-r}. \tag{36}$$

Inserting the expansions (23), (32), and (36) into (35), we express both sides of (35) in the form of a series $\sum_{r=0}^{\infty} h_r(x, t) \partial_x^{s-r}$ and equate coefficients of ∂_x^{s-r} on both sides for $s = 0, 1,..., n$ and $r = 0, 1, 2,...$. The presence of the operator ∂_x, necessitates the performance of repeated integrations by part, as well as differentiations, in order to bring the left side of (35) into the required form. Consequently, Erdelyi's [2] results cannot be applied directly. We assume, as in Section 2, that the functions $u(x, t)$ which enter in this problem, vanish identically for all $x < x_0$, for some x_0, and that $-\infty$ is the lower limit for the integral operator ∂_x^{-1}. As a result, all constants of integration are discarded when integrating by parts. The operator L is again assumed to be of hyperbolic or of mixed type.

Carrying out the necessary operations, we obtain for $r = 0$ and $s = 0, 1,..., n$, Eq. (31). This occurs because the principal parts of L_1 and L_2 were constructed so that their characteristic equations correspond to $\Omega_1(x, t, \omega) = 0$ and $\Omega_2(x, t, \omega) = 0$, (i.e., (29) and (30)), respectively. The equations for $r > 0$ can be put into the form

$$\sum_{i+j=s} [q_{i0}(x, t)\, \alpha_{jr}(x, t) + \alpha_{j0}(x, t)\, q_{ir}(x, t)] = p_{sr}(x, t) + F_{sr}(x, t),$$
$$s = 1, 2,..., n, \qquad r = 1, 2,..., \tag{37}$$

where the $F_{sr}(x, t)$ contain terms involving $q_{i\ell}(x, t)$ and its derivatives, and $\alpha_{j\ell}(x, t)$, but with the index $\ell < r$. For each $r > 0$ in (37), there are n linear equations in the $k + m = n$ unknowns $\alpha_{1r}, ..., \alpha_{kr}, q_{1r}, ..., q_{mr}$. The coefficients q_{i0} and α_{j0} in this system of equations are also the coefficients in the two relatively prime polynomials $\Omega_1(x, t, \omega)$ and $\Omega_2(x, t, \omega)$. Consequently, the determinant of the coefficients in (37) for each $r > 0$ is not zero and a unique solution for the α_{jr} and q_{ir} is obtained, when (37) is solved recursively for $r = 1, 2, 3,...$. In fact, (37) is identical in form to an equation obtained in Erdelyi's [2] work, and the nonvanishing of the determinant of the coefficients is proven there.

The specification of the α_{jr} and q_{ir} accomplishes the formal factorization of the operator L in the form $L = L_2 L_1$ as given in (32). Thus, a solution of $L_1[u] = 0$, formally satisfies the full equation $L[u] = 0$. Similarly, we can construct differential operators $\tilde{L}_1$ and $\tilde{L}_2$ such that

$$L = \tilde{L}_1 \tilde{L}_2 = \left[\sum_{i=0}^{m} \tilde{q}_i(x, t, \partial_x)\, \partial_x{}^i \partial_t^{m-i}\right] \left[\sum_{j=0}^{k} \tilde{\alpha}_j(x, t, \partial_x)\, \partial_x{}^j \partial_x^{k-j}\right], \tag{38}$$

where $\tilde{\alpha}_j$ and $\tilde{q}_i$ have expansions of the form (33) and (34) and $\tilde{\alpha}_{j0}(x, t) = \alpha_{j0}(x, t)$ and $\tilde{q}_{i0}(x, t) = q_{i0}(x, t)$. That is, the principal parts of $\tilde{L}_1$ and $\tilde{L}_2$ are identical to those of L_1 and L_2, respectively. Then a solution $v(x, t)$ of $\tilde{L}_2[v] = 0$, formally satisfies $L[v] = 0$. If either of the polynomials Ω_1 or Ω_2 admits further mutually prime factorizations, the above process can again be applied to the operator L_1 or $\tilde{L}_2$. In this way, an irreducible set of formal operators L_i can be constructed, such that the characteristics of the L_i are characteristics of L and the characteristic polynomials Ω_i of the L_i are mutually prime factors in a factorization of the characteristic polynomial Ω of L. A solution $u(x, t)$ of each of the equations $L_i[u] = 0$, formally satisfies $L[u] = 0$. If comparison or related equations can be constructed for the equations $L_i[u] = 0$, a further simplification of the theory can be achieved, and this will be seen from the results of the following section for certain classes of coincidence patterns for the characteristics of (22). Also, the general form of the operators L_i and L_2 will become meaningful, in terms of the solutions to be considered. Apart from Section 6, we shall restrict our discussion in the following sections to systems of equations, but similar results are valid for the equations considered in this section, as well as for single higher-order equations in more than two variables.

4. Progressing Wave Expansions

Formal solutions of (1) in the form

$$u(x, t) = \sum_{r=0}^{\infty} \ell_r(\varphi)\, w_r(x, t), \tag{39}$$

where $\varphi(x, t) =$ constant, is a characteristic element of the given system or equation, are generally referred to as progressing wave expansions. The $\ell_r(\varphi)$ satisfy the recursive relation

$$\frac{d}{d\varphi}\,[\ell_r] = \ell_{r-1}; \qquad r \geqslant 0, \tag{40}$$

and $\ell_0(\varphi)$ is an arbitrary function or a distribution. When φ is a real simple (or under certain additional conditions, uniformly multiple) characteristic element, in the domain D under consideration for the given problem, the functions $w_r(x, t)$ can be specified as nonsingular solutions of a recursive system of ordinary differential equations [1], [17]. These equations are obtained by inserting (39) into (1) and putting the coefficients of the $\ell_r(\varphi)$ $[r \geqslant 0]$ equal to zero. However, when the characteristics have variable multiplicity in D, the standard expansion form (39) is generally not valid, and different uniform progressing wave expansions must be constructed for each of the variety of coincidence patterns for the characteristics that can occur.

When two characteristic elements for (1) are of variable multiplicity and have coincidence patterns equivalent to those considered in Zauderer [15], the uniform progressing wave expansions constructed there can be used for the problems considered here. Indeed, let the roots of the polynomial $p_1(x, t)$ in (4), which is taken to be of degree two, correspond to the aforementioned multiple characteristics of the system (1). The reduced system in (19) which corresponds to the factor p_1 in (4), can be written as

$$\frac{\partial \hat{v}}{\partial t} + A_{11}(x, t) \frac{\partial \hat{v}}{\partial x} + \sum_{r=0}^{\infty} C_{r+1}^{11}(x, t)\, \partial_x^{-r}[\hat{v}] = 0. \tag{41}$$

where $\hat{v}(x, t)$ is a 2-vector and A_{11} and C_r^{11} $[r \geqslant 1]$ are two by two matrices. Assuming the same smoothness properties for the coefficients in (1) as those required in our previous work [15], there exist transformations (which can, in fact, be applied directly to (1)), which bring the principal part of (41) into the same canonical form as was obtained there [15]. Then, the uniform progressing wave expansions obtained previously [15] for systems of two first-order equations in two independent variables, can be constructed for (41). Given the uniform expansion for (41), we obtain the corresponding expansion for (1) via the equation $u(x, t) = P(x, t, \partial_x)\, v(x, t)$ where the n-vector $v(x, t)$ has the components of $\hat{v}(x, t)$ as the first two and only nonzero components. However, once the general form of the uniform expansion for a given coincidence pattern is known, its terms are most easily specified by inserting the assumed expansion directly into (1) and specifying its terms recursively.

An interesting feature of the present construction, is the presence of iterated integral terms in (41). Such terms were shown to be necessary in the construction of comparison equations (in terms of whose solutions the uniform expansions are given) in our previous work [15], even though such terms did not occur explicitly in the given equations considered there. Thus, from the more general problem considered here, it is seen that the forms taken by the aforementioned comparison equations are not unexpected.

For coincidence patterns of characteristic elements involving more than two characteristics or, in general, differing from those considered above, no general results beyond those obtained from the simplification and factorization methods presented in the previous sections are available. Once comparison equations for each of these cases are constructed, a further simplification of the asymptotic theory for the given equations in terms of series involving solutions of the comparison equations can be achieved. This parallels the development of the asymptotic theory of ordinary differential equations with a large parameter [8, 13].

When $\varphi =$ constant is a real simple characteristic for (1), we may assume the linear factor $p_1(x, t, \lambda)$ in (4) corresponds to it and obtain from (19) the reduced equation

$$\frac{\partial v^1}{\partial t} + A_{11}(x, t) \frac{\partial v^1}{\partial x} + \sum_{r=0}^{\infty} C_{r+1}^{11}(x, t)\, \partial_x^{-r}[v^1] = 0, \tag{42}$$

where $v^1(x, t)$ is the first component of the n-vector $v(x, t)$, and A_{11} and C_r^{11} are scalar functions. Putting $v(x, t) = v^1(x, t)\, e_1$, where the vector e_1 has 1 as its first and only nonzero component, we obtain for the solution $u(x, t)$ of (1),

$$u(x, t) = P(x, t, \partial_x)\, v = v^1(x, t)\, e_1 + \sum_{r=1}^{\infty} \partial_x^{-r}[v^1(x, t)]\, P_r(x, t)\, e_1 . \tag{43}$$

The matrices P_r are defined in Section 2 [see (20) and (21)] and their special form implies that e_1 and $P_r e_1$ ($r \geqslant 1$) are mutually orthogonal.

By a change of independent variables in (1), the characteristic curves $\varphi =$ constant can be transformed into the lines $x =$ constant. Setting

$$v^1(x, t) = \ell_0(x)\, w(x, t), \tag{44}$$

where ℓ_0 is defined as above, we obtain from (43), after repeated integrations by part,

$$u(x, t) = \ell_0(x)\, w(x, t)\, e_1 + \sum_{r=1}^{\infty} \ell_r(x)\, \sigma_r(x, t). \tag{45}$$

The function $w(x, t)$ satisfies the equation

$$\ell_0(x)\, w_t + \sum_{r=0}^{\infty} C_{r+1}^{11}(x, t)\, \partial_x^{-r}[\ell_0(x)\, w] = \ell_0(x)\, w_t + \sum_{r=0}^{\infty} \ell_r(x)\, N_r[w] = 0, \tag{46}$$

where the N_r are differential operators in x, whose order increases with r, obtained on integrating by parts in the terms $\partial_x^{-r}[\ell_0(x)\, w]$. The $\sigma_r(x, t)$ in (45) are all given in terms of $w(x, t)$. The series (45) represents a decomposition of the solution vector $u(x, t)$ into a sum of mutually orthogonal vectors i.e., $\ell_0 w e_1$ and $\sum \ell_r \sigma_r$. Each of these vectors lies in an invariant subspace associated with the block-diagonalized principal part of (1) given in Section 2.

The progressing wave solution (39) can be retrieved from (45) and (46) by expanding $v^1 = \ell_0 w$ in the form (39) and solving for the expansion coefficients w_r recursively from (46). Consequently, our result (45) and (46) may be expected to have wider validity than (39). For example, if we let $\ell_0(x)$ be a rapidly oscillating exponential,

$$\ell_0(x) = \exp[ikx], \qquad \ell_r(x) = [ik]^{-r} \exp[ikx], \tag{47}$$

where k is a large parameter, Eq. (46) for $w(x, t)$ takes the form

$$w_t + \sum_{r=0}^{\infty} [ik]^{-r} N_r[w] = 0. \tag{48}$$

Assuming (1) has constant coefficient matrices A and B, the expansion (39) may contain secular terms $w_r(x, t)$, i.e., terms which grow algebraically in t, as t increases. Consequently, for large t, higher-order terms in the expansion (39) will be of the same order as lower-order terms, so that the expansion is difficult to interpret. This will be shown explicitly in Section 6, where it will be seen how (48) can be used to remove this inadequacy of (39). Now, (48) is a parabolic equation in which the order of the differential operators M_r increases unboundedly with r. However, approximate solutions of (48) and, thereby of (1) can be obtained by truncating (48) at some power of k, as will be shown for the special case considered in Section 6.

In the following section we present a direct method for constructing solutions of the general form (45) and (46) for hyperbolic systems in more than two variables. Now, (46) may be referred to as the reduced equation for the given system (1), relative to a simple characteristic element. Consequently, when all the characteristics are simple (or uniformly multiple), constructing solutions of a hyperbolic system in the above form leads to a reduction of the system in terms of equations corresponding to (46).

5. Formal Simplification of Linear Hyperbolic Systems

We consider the first-order linear hyperbolic system

$$M[u] = u_t + \sum_{i=1}^{m} A^i(x, t)\, u_{x_i} + B(x, t)\, u = 0, \tag{49}$$

where $u(x, t)$ is an n-vector function, x is an m-vector, and the n by n matrices A^i $[i = 1,..., m]$ and B depend smoothly on x and t. A progressing wave solution for (49),

$$u(x, t) = \ell_0(\varphi)\, w_0(x, t) + \sum_{r=1}^{\infty} \ell_r(\varphi)\, w_r(x, t) \tag{50}$$

can be constructed for characteristic elements $\varphi(x, t) =$ constant which satisfy the equation

$$\det\left[I\varphi_t + \sum_{i=1}^{m} A^i \varphi_{x_i}\right] \equiv \det A = 0. \tag{51}$$

and are simple or have constant multiplicity. In the latter case, an additional condition, to be given below, should be satisfied. Inserting (50) into (49) we obtain, on using (40),

$$M[u] = \ell_{-1}(\varphi)\, Aw_0 + \sum_{r=0}^{\infty} \{Aw_{r+1} + M[w_r]\}\, \ell_r(\varphi) = 0. \tag{52}$$

Solving for the w_r recursively by equating coefficients of the $\ell_r(\varphi)$ to zero, yields the standard progressing wave expansion for (49). Instead, we construct a solution equivalent in form to that given in the previous section, i.e., Eqs. (45) and (46).

For a given characteristic element $\varphi =$ constant, we decompose the space R_n into a direct sum of two subspaces which are invariant with respect to the characteristic matrix $A(x, t)$ and correspond to the null and nonnull eigenvalues of A. This decomposition can be accomplished by the use of projection operators P and Q which map n-vectors into the aforementioned subspaces, respectively. These operators can be represented as n by n matrices having the properties

$$\begin{aligned} P^2 = P, \qquad Q^2 = Q, \qquad P + Q = I \\ PA = AP, \qquad QA = AQ, \qquad PQ = QP = 0. \end{aligned} \tag{53}$$

For example, suppose $\varphi =$ constant is a simple characteristic, A a symmetric matrix, and R its unit nullvector. Then, the matrices

$$P = RR^T, \qquad Q = I - RR^T, \tag{54}$$

where R^T is the transpose of the column vector R, are the required projection operators. Similarly, if $\varphi =$ constant is a characteristic of uniform multiplicity s, A is a symmetric matrix, and $R_1, R_2, \ldots, R_s$ are its mutually orthogonal unit null vectors, the required matrices are

$$P = \sum_{i=1}^{s} R_i R_i^T, \qquad Q = I - \sum_{i=1}^{s} R_i R_i^T. \tag{55}$$

The construction of P and Q in the general case is well known.

Following the result of the previous section, we decompose the solution $u(x, t)$ into a sum of vectors lying in the invariant subspaces of A considered above. That is, we specify the vectors $w_r(x, t)$ such that

$$Pw_0 = w_0, \qquad Qw_r = w_r, \qquad r \geqslant 1. \tag{56}$$

Projecting in (52) with P we obtain

$$PM[u] = \sum_{r=0}^{\infty} \ell_r(\varphi)\, PM[w_r] = 0, \tag{57}$$

on using (53) and noting that $Aw_0 = 0$ and $PAw_{r+1} = APw_{r+1} = 0$. Projecting with Q in (52) gives

$$QM[u] = \sum_{r=0}^{\infty} \ell_r(\varphi)\,\{Aw_{r+1} + QM(w_r]\} = 0 \tag{58}$$

again using (53). Putting the coefficients of the $\ell_r(\varphi)$ equal to zero in (58), leads to a recursive system for the $w_r(x, t)$,

$$Aw_{r+1} = -QM[w_r]; \qquad r \geqslant 0. \tag{59}$$

In view of (53) and (56), the w_r $[r \geqslant 1]$ are uniquely specified in terms of $w_0(x, t)$ and its derivatives. Inserting the resulting expressions for $w_r(x, t)$ $[r \geqslant 1]$ into (57), we obtain a reduced equation for $w_0(x, t)$, which corresponds to (46). It is readily seen that the higher terms in the expansion (57) contain increasingly higher derivatives of $w_0(x, t)$.

When all the characteristics of (49) are simple, n uncoupled reduced equations of the form (57) can be constructed. In terms of these equations n linearly independent solutions of the form (50) can be constructed. In this sense, the above result may be said to yield a simplification of the given system (49). A similar result obtains if $\varphi =$ constant is a characteristic of constant multiplicity s and the corresponding characteristic matrix A has s linearly independent nullvectors. Then, a reduced system can be constructed, corresponding to the subspace spanned by the s nullvectors.

As was seen above, the standard progressing wave expansion can be retrieved from our result. Since the reduced equation (57) has a rather complicated form, the present results should be used in obtaining the solution of (49) only when inadequacies in the standard progressing wave expansions arise, such as the presence of secular terms which will be considered in the following section, for the Klein–Gordon equation.

6. The Klein–Gordon Equation

To illustrate the methods presented in the previous sections, we shall apply them to the one-dimensional Klein–Gordon equation

$$U_{tt} - U_{xx} + U = 0, \tag{60}$$

which may also be represented as the first-order system

$$\begin{aligned} U_t + U_x - V &= 0 \\ V_t - V_x + U &= 0. \end{aligned} \tag{61}$$

While the presence of constant coefficients in the above equations simplifies the calculations to be carried out below, it leads to secular terms in the standard progressing wave expansions for (60) and (61). It will be shown that on using instead the expansion forms given in the previous sections, results free of secularities can be obtained.

We consider first a progressing wave expansion for (60) and expand $U(x, t)$ as in (39), with $\varphi(x, t) = x - t$, which is a characteristic for (60). [A similar result obtains for the second characteristic $\varphi = x + t$.] Inserting the expansion (39) into (60) and equating the coefficients of $\ell_{-1}(\varphi)$ and $\ell_0(\varphi)$ to zero, yields

$$\frac{\partial w_0}{\partial t} + \frac{\partial w_0}{\partial x} = 0 \tag{62}$$

$$\frac{\partial w_1}{\partial t} + \frac{\partial w_1}{\partial x} = \frac{1}{2}\left[\frac{\partial^2 w_0}{\partial t^2} - \frac{\partial^2 w_0}{\partial x^2} + w_0\right] \tag{63}$$

so that

$$w_0(x, t) = \hat{w}_0(x - t); \qquad w_1(x, t) = \hat{w}_1(x - t) + (t/2)\,\hat{w}_0(x - t), \tag{64}$$

where $\hat{w}_0$ and $\hat{w}_1$ are arbitrary functions. The secularity is exhibited in the term $(t/2)\,\hat{w}_0(x - t)$. Thus, if we set $\ell_0(x - t) = \exp[ik(x - t)]$, (as in (47)) where k is a large parameter, it is seen that for $t = O(k)$, the second term w_1 in the expansion of U, [which now proceeds in inverse powers of k], is of the same order of magnitude as the leading term w_0. Additional secular terms occur in each further term of the expansion but will not be considered here.

Similarly, on writing (61) in the form

$$L[u] = u_t + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} u_x + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} u = 0, \tag{65}$$

where the vector u has components U and V, expanding $u(x, t)$ as in (39), again with $\varphi = x - t$, and proceeding as above, we find that w_0 has the form

$$w_0(x, t) = \sigma_0(x, t) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{66}$$

where $\sigma_0(x, t)$ is a scalar function. The terms $w_r(x, t)$ $[r \geqslant 1]$ satisfy the equations

$$\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} w_{r+1} = -L[w_r]; \qquad r \geqslant 0. \tag{67}$$

The singularity of the coefficient matrix requires the right side of (67) to satisfy a compatibility condition. For $r = 0$, this yields

$$\frac{\partial \sigma_0}{\partial t} + \frac{\partial \sigma_0}{\partial x} = 0, \tag{68}$$

so that $\sigma_0 = \hat{\sigma}_0(x - t)$. Since $w_1(x, t)$ has the form

$$w_1(x, t) = \sigma_1(x, t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{\hat{\sigma}_0}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{69}$$

The compatibility condition for $r = 1$ gives

$$\frac{\partial \sigma_1}{\partial t} + \frac{\partial \sigma_1}{\partial x} = \frac{1}{2} \hat{\sigma}_0(x - t). \tag{70}$$

Therefore, secularity again results in the form

$$\sigma_1(x, t) = \hat{\sigma}_1(x - t) + (t/2)\, \hat{\sigma}_0(x - t), \tag{71}$$

with additional secular terms occurring in each further term in the expansion of $u(x, t)$.

We now apply the methods of the previous sections to (60) and (65) and show how they can be used to construct expansions free of secularity. We begin with the simplification method of Section 2 as applied to (65). Since the coefficient matrix of u_x in (6) is already in diagonal form, the expansion form (11) can be used directly. The matrices P_r and C_r in (11) and (19), respectively, are to be specified, and this task is considerably simplified since the coefficient matrices in (65) are constant. Applying the results of Section 2, it is readily seen that C_1 is the zero matrix and

$$C_2 = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}; \qquad P_1 = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}. \tag{72}$$

The system (19) takes the form

$$v_t + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} v_x + \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \partial_x^{-1}[v] + \cdots = 0, \tag{73}$$

on retaining only the leading terms.

Before examining (73) more closely, we apply the factorization method of Section 3 to Eq. (60). The procedure is straightforward, since (60) has constant coefficients and gives

$$L_1 L_2 U = (\partial_t - \partial_x + \tfrac{1}{2}\partial_x^{-1} + \cdots)(\partial_t + \partial_x - \tfrac{1}{2}\partial_x^{-1} + \cdots) U = 0, \tag{74}$$

where the operators L_1 and L_2 commute. The equations resulting from each of the factors in (74) are identical to those satisfied by the components of the vector $u(x, t)$ in (73), up to the number of terms retained above.

Returning to our discussion of (73), we show how the procedure described in Section 4 as applied to (73) yields a result where the secularity is absent. With

$$v(x, t) = v^1(x, t) \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

we obtain

$$\frac{\partial v^1}{\partial t} + \frac{\partial v^1}{\partial x} - \frac{1}{2} \partial_x^{-1}[v^1] + \cdots = 0, \tag{75}$$

which corresponds to (42). Then, from (43) we have

$$u(x, t) = Pv(x, t) = v^1(x, t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \tfrac{1}{2}\partial_x^{-1}[v^1] \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \cdots \tag{76}$$

on using (72). On expanding $v^1(x, t)$ in progressing wave form as in (39), and integrating by parts in the term $\partial_x^{-1}[v^1]$, it is easily seen that the results (68)–(71) are obtained. Instead we express $v^1(x, t)$ in the form (44),

$$v^1(x, t) = \ell_0(x - t)\, w(x, t), \tag{77}$$

and obtain the equation for $w(x, t)$,

$$\ell_0(x - t) \left[\frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} \right] - \frac{1}{2} \ell_1(x - t)\, w + \cdots = 0, \tag{78}$$

on inserting (77) into (75), itegrating by parts, and retaining only the leading terms. Putting $\ell_0(x - t) = \exp[ik(x - t)]$ in (78) we have

$$\frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} - \frac{1}{2ik} w = O[k^{-2}]. \tag{79}$$

Truncating and retaining only the indicated terms in (79) gives

$$w(x, t) = \exp[t/2ik]\, \hat{w}(x - t), \tag{80}$$

where $\hat{w}$ is an arbitrary function. Inserting (80) into (76), integrating by parts and retaining only the leading terms gives

$$u(x, t) = \exp\left[ik(x - t) + \frac{t}{2ik}\right] \left\{\begin{bmatrix}1\\0\end{bmatrix} + \frac{1}{2ik}\begin{bmatrix}0\\1\end{bmatrix}\right\} \hat{w}(x - t) + O[k^{-2}]. \tag{81}$$

Clearly, the secular term obtained in (71) is a consequence of expanding out the exponential in (80), while the above result is free of secularity. Retaining additional terms in (79) removes secular terms arising in further terms of the expansion of $u(x, t)$. We do not consider this any further nor do we examine this problem when $\ell_0(x - t)$ has a different form, say, if it is a distribution.

With regard to the scalar Klein–Gordon equation (60), we consider the equation $L_2 U = 0$, (see (74)) and express $U(x, t)$ as in (77). Proceeding exactly as above it is easily seen that U has the form (80). This term corresponds to first two terms in the expansion of $U(x, t)$ considered above (see (64)) except that secularity is absent. We note that terms corresponding to the arbitrary functions $\hat{w}_1$ and $\hat{\sigma}_1$ in (64) and (71), respectively, can be obtained in our result simply by adding to the above solutions a second solution of identical form whose leading term is $O[k^{-1}]$.

We now show that the method of Section 5 leads to a result identical with that obtained above for (65) on using the simplification method of Section 2. With $\varphi = x - t$ in (50), the matrix corresponding to A in (51) is for our problem

$$A = \begin{bmatrix}0 & 0\\0 & 2\end{bmatrix}. \tag{82}$$

Since A is symmetric, the projection operators P and Q can be formed in terms of the unit nullvector R of A. Thus

$$R = \begin{bmatrix}1\\0\end{bmatrix}; \quad P = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}; \quad Q = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \tag{83}$$

where P and Q were constructed using (54). From (56) we find that

$$w_0(x, t) = W_0(x, t)\begin{bmatrix}1\\0\end{bmatrix}, \quad w_r(x, t) = W_r(x, t)\begin{bmatrix}0\\1\end{bmatrix}; \quad r \geqslant 1, \tag{84}$$

where the $W_r(x, t)$ are scalar functions. From (57) we have

$$PL[u] = \ell_0(x-t)\left(\frac{\partial W_0}{\partial t} - \frac{\partial W_0}{\partial x}\right) - \sum_{r=1}^{\infty} \ell_r(x-t)\, W_r(x, t) = 0. \tag{85}$$

Using (59) with $r = 0$, we obtain

$$W_1(x, t) = \tfrac{1}{2}\, W_0(x, t). \tag{86}$$

Inserting (86) into (85) we see that, to the number of terms indicated, the resulting equation agrees with (78).

We conclude our discussion of the Klein–Gordon equation by presenting a method for constructing the reduced equation $L_2 U = 0$ (see (74)), without having to factor the differential operator as was done above. In this sense, it is related to the method of Section 5, where a reduced equation for the given characteristic was constructed directly. For simplicity, we set in (60),

$$U(x, t) = \exp[ik(x-t)] \sum_{r=0}^{\infty} V_r(x, t)\,(ik)^{-r}, \tag{87}$$

rather than using the general expansion form (39) which involves the $\ell_r(x-t)$. The leading terms in the resulting expansion are

$$\begin{aligned} -2ik\left[\frac{\partial V_0}{\partial t} + \frac{\partial V_0}{\partial x}\right] + \left[\frac{\partial^2 V_0}{\partial t^2} - \frac{\partial^2 V_0}{\partial x^2} + V_0 - 2\left(\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x}\right)\right] + \cdots \\ = 0. \end{aligned} \tag{88}$$

Solving by equating the bracketed terms in (88) to zero leads to secularity as was shown at the beginning of this section. Since the term V_0 in the second bracket accounts for the secular term in the conventional method, (compare (63) and (64)), we reorder the terms in (88) as follows,

$$\begin{aligned} \left\{-2ik\left(\frac{\partial V_0}{\partial t} + \frac{\partial V_0}{\partial x}\right) + V_0\right\} + \left\{\frac{\partial^2 V_0}{\partial t^2} - \frac{\partial^2 V_0}{\partial x^2} - 2\left(\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x}\right)\right\} + \cdots \\ = 0, \end{aligned} \tag{89}$$

and treat the first bracket in (89) as the leading term. Equating the first bracket to zero gives

$$\frac{\partial V_0}{\partial t} + \frac{\partial V_0}{\partial x} - \frac{1}{2ik}\, V_0 = 0, \tag{90}$$

whose solution is of the form given in (80). Since for V_0 satisfying (90) we have

$$\frac{\partial^2 V_0}{\partial t^2} - \frac{\partial^2 V_0}{\partial x^2} = \frac{1}{2ik}\left(\frac{\partial V_0}{\partial t} - \frac{\partial V_0}{\partial x}\right), \tag{91}$$

This means that the above term must be shifted to the next (unspecified) term in (89), which is $O(k^{-1})$. Now putting $V_1(x, t) \equiv 0$, we see that (89) is equivalent to (79). Were $V_1(x, t)$ not equated to zero, an argument similar to the above would be required to remove a secular term involving $V_1(x, t)$. Again, since all equations occurring in (89) have constant coefficients, the term (91) results in a secular term in the $O(k^{-1})$ equation of (89). This can be removed by again reordering terms in (89) and replacing the leading equation (90) by

$$\frac{\partial V_0}{\partial t} + \frac{\partial V_0}{\partial x} - \frac{1}{2ik} V_0 + \frac{1}{4k^2}\left(\frac{\partial V_0}{\partial t} - \frac{\partial V_0}{\partial x}\right) = 0. \tag{92}$$

To the order of k retained, since

$$\frac{\partial V_0}{\partial t} = -\frac{\partial V_0}{\partial x} + O(k^{-1}) \tag{93}$$

we have, instead of (92),

$$\frac{\partial V_0}{\partial t} + \frac{\partial V_0}{\partial x} - \frac{1}{2ik} V_0 - \frac{1}{2k^2}\frac{\partial V_0}{\partial x} = 0. \tag{94}$$

This equation would also result if we had carried an additional term in the factor L_2 in (74) and applied the method given earlier. The effect of the present method is to permit all the $V_r(x, t)$ with $r \geqslant 1$ to be eliminated from the expansion (87) so that $U(x, t) = \exp[ik(x - t)]\ V_0(x, t; k)$. The resulting full equation for V_0 involves an infinite series in inverse powers of k, whose leading terms are given in (94). The solution is thus of the form (77) and the present result is identical with that obtained via factorization. It is also readily seen that on using this solution form (i.e., with the $V_r \equiv 0$, $(r \geqslant 1)$ in (87)), Eq. (94) and all higher-order equations for V_0 can be derived by applying an iteration procedure in (88). The steps in the iteration process leading to (94) are contained in (90), (91), and (93).

7. Conclusion

Using techniques patterned after those developed in the asymptotic theory for ordinary differential equations with a large parameter, we have presented a formal simplification and factorization theory for linear partial differential equations in two independent variables. We have seen that appropriate solution forms for the reduced systems and equations obtained are progressing wave expansions. Thereby, certain results obtained previously in the construction of uniform progressing wave expansions are extended and clarified. Applications of some of these results to the asymptotic solution of initial and boundary value problems for linear partial differential equations, have been indicated elsewhere [14–16]. Additionally, a new expansion form for solutions of hyperbolic equations was considered and shown to be useful in dealing with difficulties arising in conventional progressing wave expansions which contain secular terms.

Essentially, the simplification method presented above, achieves a decomposition of a system of equations

$$L[u] + N[u] = 0, \tag{95}$$

where L and N are matrices or matrix differential operators, when L itself admits such a decomposition. The decomposition of the full system is accomplished by means of a series expansion. In the problem considered in Section 2, L is the principal part of the given differential operator (1) while in Section 5, L is essentially the characteristic matrix A.

A similar concept is used in the Chapman–Enskog theory for Boltzmann's equation [5, 10]; in bifurcation theory for nonlinear eigenvalue problems (see the survey article by Berger in Keller–Antman [6]); and in the method of smoothing expounded by Frisch in the theory of wave propagation in random media [4]. These methods were developed for the purpose of extending the validity of standard perturbation expansions which had proved inadequate for certain aspects of the problems considered. More specific applications of our results and extensions of the simplification method to various nonlinear problems will be presented elsewhere.

Acknowledgment

This work was begun while the author was on leave at the Weizmann Institute of Science, Rehovot, Israel. The support of the Applied Mathematics Department is gratefully acknowledged.

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